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2004 J. Phys. A: Math. Gen. 37 4821

(http://iopscience.iop.org/0305-4470/37/17/012)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 4821–4842 PII: S0305-4470(04)75444-X

Poisson structures for reduced non-holonomic systems

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Received 30 January 2004
Published 14 April 2004
Online at stacks.iop.org/JPhysA/37/4821 (DOI: 10.1088/0305-4470/37/17/012)

Abstract

Borisov, Mamaev and Kilin have recently found certain Poisson structures with respect to which the reduced and rescaled systems of certain non-holonomic problems, involving rolling bodies without slipping, become Hamiltonian, the Hamiltonian function being the reduced energy. We study further the algebraic origin of these Poisson structures, showing that they are of rank 2 and therefore the mentioned rescaling is not necessary. We show that they are determined, up to a non-vanishing factor function, by the existence of a system of first-order differential equations providing two integrals of motion. We generalize the form of the Poisson structures and extend their domain of definition. We apply the theory to the rolling disc, the Routh's sphere, the ball rolling on a surface of revolution, and its special case of a ball rolling inside a cylinder.

PACS numbers: 02.40.k, 03.04.t
Mathematics Subject Classification: 70G45, 70E18, 70F25

1. Introduction

In recent years there has been an increasing interest in the geometric treatment of non-holonomic mechanical systems (see, e.g., [2, 3, 5, 6, 10, 12, 19, 28, 29, 31, 38–40]). In particular, it has been recognized that the Hamiltonian formulation of such systems can be stated in terms of an *almost-Poisson* bracket, that is, a biderivation of functions of phase space, antisymmetric in its arguments but which does not necessarily fulfil the Jacobi identity (see, e.g., [1, 11, 36]). Therefore, for researchers in this field, it seems to be usual for the conceptual association of the Hamiltonian formulation of non-holonomic mechanical systems with almost-Poisson structures.

On the other hand, there exist non-holonomic systems which, after certain reductions are performed, admit a Hamiltonian formulation after a 'rescaling of time' is carried out, by means of rescaling factors (sometimes called *invariant measures*) of the reduced vector field of the system. This is the case for the so-called LR systems, which are systems formulated on compact Lie groups endowed with a left-invariant metric and right-invariant non-holonomic constraints. After a rescaling of time, their corresponding reduced systems become integrable Hamiltonian systems describing geodesic flows on unit spheres [24]. In [9], a necessary and sufficient condition for the existence of an invariant measure for the reduced dynamics of generalized Chaplygin systems of mechanical type is given. Another recent work on this line is [43]. For a classic treatment of the theory of Chaplygin's reducing multiplier, see section III-12 of [33]. Thus, it could be conceptually associated as well the existence of specific rescaling factors for these reduced systems with the possibility of formulating them in a Hamiltonian way.

In addition, Borisov, Mamaev and Kilin [7, 8] have recently found a Poisson structure for each studied case of reduced non-holonomic systems, such that the reduced system becomes Hamiltonian, with respect to such a structure, after a rescaling, the Hamiltonian function being the reduced energy. The examples treated by them are classical in the literature, consisting mainly of rolling bodies without slipping, namely a rigid body of revolution rolling on a plane, in particular the *Routh's sphere* (see section 4.2), the rolling disc (to be treated in section 4.1), the motion of a homogeneous ball on a surface of revolution (see section 4.3), which is sometimes called *Routh's problem* [42, 43], and other cases, like axisymmetric bodies rolling on a plane and a sphere [7]. There is a strong emphasis in [7, 8] in the sense that the Poisson structure for each case can be found *after a rescaling of time* of the reduced vector field.

Our primary motivation for this work was to understand the origin of the two integrals of motion appearing in the mentioned problem of a ball rolling without slipping inside a surface of revolution, which are not given, in general, in an explicit form but being related to the solutions of a system of first-order non-autonomous differential equations [25, 35, 42]. This also happens in the other mentioned cases. The results of [7, 8] suggest that such systems can be interpreted as the equations providing a set of functionally independent Casimir functions of the Poisson structure they find for each specific case. Therefore, it seemed to be worth investigating further such Poisson structures, in particular to clarify their domain of definition and basic properties. Let us note that another recent approach, devoted to the study of Poisson structures which can be associated with never-vanishing vector fields on manifolds of arbitrary dimension $d \geqslant 2$, with *fibrating periodic flows*, is given in [23].

It follows that the previously mentioned Poisson structures have a rather peculiar form. In particular, the associated characteristic distributions have rank 2 in the open sets of the reduced spaces considered in [7, 8]. This property implies that such Poisson structures, when multiplied by a never vanishing function, are again Poisson structures of the same type. The immediate consequence is that the above-mentioned reduced non-holonomic systems are already Hamiltonian with respect to one of these Poisson structures without any need of rescaling.

Other interesting result is that, in the cases studied, the Poisson structures obtained can be extended from their original domains of definition, namely (open sets of) semialgebraic subvarieties of $\mathbb{R}^5$, to an open set of the ambient space. Such extended Poisson structures become zero only at the so-called *singular equilibria* of the reduced systems. Moreover, the existence of these (extended) Poison structures, from an algebraic point of view, is only caused by the existence of integrals of motion of the reduced vector field related to the solutions of the mentioned systems of first-order differential equations.

This paper is organized as follows. In section 2, we briefly review some notions of Poisson geometry and in particular, of Poisson structures of rank 2. In section 3, we show the explicit expressions of certain bivectors in $\mathbb{R}^4$ and $\mathbb{R}^5$, determined up to a non-vanishing factor function, by choosing the 1-forms in their kernels to have a specific form, and we prove that they are in fact Poisson bivectors of rank 2. Section 4 is devoted to show the application of the previous results in specific examples, namely, the rolling disc, the Routh's sphere, and the ball rolling on a surface of revolution. We will use the formulation of [17, 18] and [25], respectively, of these problems, rather than that of [7, 8]. However, we point out the equivalence of both treatments in the last case. We also treat the special case of a ball rolling inside a cylinder. Finally, we end with some conclusions and an outlook for further research.

2. On Poisson structures of rank 2

For the sake of completeness and in order to fix some notations, we will recall some well-known notions on Poisson manifolds, and in particular, we will focus on Poisson structures of rank 2. For more details see, e.g., [26].

Given a differentiable manifold M, a *Poisson structure* on M is defined by an antisymmetric bilinear map $\{\cdot, \cdot\}$ which is a derivation on both of its arguments, satisfying moreover the Jacobi identity. A manifold M endowed with a Poisson structure is called a *Poisson manifold*.

Thus, it is possible to associate with each function f a unique vector field X_f such that, for any other function g, we have $X_f g = \{f, g\}$. The vector field X_f is called *Hamiltonian vector field* associated with the *Hamiltonian function* f. This association defines an homomorphism of the Lie algebra $(C^\infty(M), \{\cdot, \cdot\})$ onto the Lie algebra of vector fields in M. A *Casimir function* or *Casimir* for short, is a function c such that $X_c = 0$.

Moreover, on every Poisson manifold, there exists a unique twice contravariant antisymmetric tensor field (called *bivector field* for short) Λ such that $\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g)$ for every pair of functions (f, g). This tensor field is called the *Poisson tensor* of the structure, and the manifold M, endowed with its Poisson structure, will be denoted (M, Λ). The existence of such a tensor field is due only to the antisymmetry and derivation properties of the Poisson bracket. The fulfillment of the Jacobi identity for the Poisson bracket is equivalent [27] to the vanishing of the *Schouten–Nijenhuis bracket* of Λ with itself, $[\Lambda, \Lambda] = 0$. The Schouten–Nijenhuis bracket [34, 37] is the unique extension of the Lie bracket of vector fields to the exterior algebra of multivector fields. Some of its properties are

$$\begin{aligned}
&[P, Q] = -(-1)^{(p-1)(q-1)}[Q, P] \\
&[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{(p-1)q} Q \wedge [P, R] \\
&[P \wedge R, Q] = P \wedge [R, Q] + (-1)^{(q-1)r}[P, Q] \wedge R
\end{aligned} \tag{1}$$

where P, Q, R are completely antisymmetric contravariant tensors of degree p, q, r, respectively. For more details and properties on the Schouten–Nijenhuis bracket see, e.g., [13, 30, 34, 37] and references therein.

Take a local chart of M, with domain U and associated local coordinates $(x_1, \ldots, x_n)$, where $n = \dim M$. We will denote by Λ_{ij} $(1 \leqslant i, j \leqslant n)$ the components of the Poisson tensor Λ in the previous chart. The expression of the Poisson bracket of the restriction of the two functions f, g to U, also denoted by f, g, reads

$$\{f, g\} = \Lambda_{ij} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}$$

where summation in the repeated indices is understood. In particular we have $\{x_i, x_j\} = \Lambda_{ij}$. The Poisson tensor admits the local expression

$$\Lambda = \sum_{i<j}^{n} \Lambda_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \tag{2}$$

in these coordinates.

Given a Poisson manifold (M, Λ), it can be defined the fibred morphism $\Lambda^\sharp : T^*M \to TM$ such that for any pair of 1-forms α, β, $\langle \Lambda^\sharp(\alpha), \beta\rangle = \Lambda(\alpha, \beta)$. The image of the morphism $\Lambda^\sharp$, $C = \Lambda^\sharp(T^*M)$, is called *the characteristic distribution* of the Poisson structure, and the *characteristic space* on $x \in M$ is the vectorial subspace $C_x = \Lambda^\sharp_x(T^*_xM)$ of T_xM. The *rank* of the structure on the point x is the rank of $\Lambda^\sharp_x$, i.e., the dimension of C_x. Note that the annihilator of the characteristic distribution, i.e., $C^0 = \{\beta \in \Lambda^1(M) | \Lambda(\beta, \alpha) = 0, \forall\alpha \in \Lambda^1(M)\}$, is $\ker \Lambda^\sharp$, and we have $\operatorname{rank} \Lambda^\sharp_x + \dim\ker \Lambda^\sharp_x = n$, for all $x \in M$. In general, the rank of the structure varies with x and thus C is not in general a subbundle of TM.

Consider now a Poisson manifold (M, Λ), $\dim M = n$, such that in the domain of a local chart (U, ϕ) the structure has constant rank equal to 2. Theorem 11.5 of chapter III in [26] (or corollary 2.3 in [41]) assures us that the associated local coordinates, denoted by $(x, y, z_1, \ldots, z_{n-2})$, can be chosen such that for $1 \leqslant k, l \leqslant n-2$,

$$\{y, x\} = 1 \qquad \{x, z_k\} = 0 \qquad \{y, z_k\} = 0 \qquad \{z_k, z_l\} = 0. \tag{3}$$

We are now in a position to prove a simple result, but important for our purposes here:

Proposition 1. *Let (M, Λ) be a Poisson manifold of (locally) constant rank equal to 2. Then, for each never-vanishing smooth function $a \in C^\infty(M)$, $(M, a\Lambda)$ is a Poisson manifold of (locally) constant rank equal to 2, with the same characteristic distribution.*

Proof. We have to prove that the Schouten–Nijenhuis bracket $[a\Lambda, a\Lambda]$ vanishes, the other needed properties being obvious. From the paragraph 18.8 of chapter V of [26], we have that

$$[a\Lambda, a\Lambda] = 2a\Lambda^\sharp(\mathrm{d}a) \wedge \Lambda.$$

It suffices to compute the previous expression on a coordinate neighbourhood like that described in the previous paragraph, with respect to the Poisson tensor Λ [30]. We have

$$[a\Lambda, a\Lambda](\mathrm{d}x, \mathrm{d}y, \mathrm{d}z_k) = 2a(\Lambda^\sharp(\mathrm{d}a) \wedge \Lambda)(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z_k) = 0 \qquad 1 \leqslant k \leqslant n-2$$

because z_k are Casimir functions of Λ, and $\mathrm{d}z_k$ enters at least once as argument of Λ in all terms of the previous expression. For other possible arguments, the expression vanishes by the same reason. □

Example 1. Let M be an n-dimensional manifold and X, Y two vector fields such that for all $x \in M$, the Lie bracket $[X, Y]_x$ belongs to the subspace of T_xM generated by X_x and Y_x. Then, $\Lambda = X \wedge Y$ is a Poisson tensor of rank 2 except where X and Y are linearly dependent. This is easily seen by deducing from the properties of the Schouten–Nijenhuis bracket (1) the relation $[X \wedge Y, X \wedge Y] = 2X \wedge Y \wedge [X, Y]$, see also [1, 13].

Remark. Note that it is essential in proposition 1 the assumption that the initial bivector is Poisson, which assures the existence of local coordinates satisfying (3). The existence of a bivector whose rank is always 2 is not enough to conclude that it is a Poisson bivector. A simple counter-example is the following. Take $M = \mathbb{R}^3$, with coordinates (x, y, z). Let X, Y be vector fields in the kernel of $y\mathrm{d}x - x\mathrm{d}y + \mathrm{d}z$ given by

$$X = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z} \qquad Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}.$$

Then, $\Lambda = X \wedge Y = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \wedge \frac{\partial}{\partial z}$ is an everywhere rank 2 bivector but is not Poisson, since $[X, Y] = 2\frac{\partial}{\partial z}$ and

$$[\Lambda, \Lambda] = [X \wedge Y, X \wedge Y] = 4\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}.$$

The vector fields X, Y and $[X, Y]$ in this example close on a Lie algebra isomorphic to the Heisenberg–Weyl Lie algebra $\mathfrak{h}(3)$, see, e.g., [16].

3. Some Poisson structures of rank 2 in $\mathbb{R}^4$ and $\mathbb{R}^5$

We will construct in this section some Poisson structures of rank 2 in $\mathbb{R}^4$ and $\mathbb{R}^5$ by imposing that the kernel of the corresponding bivectors consists of a set of two and three specific 1-forms, respectively. Such 1-forms will determine codistributions which are integrable in the sense of Frobenius. We will prove that the resulting bivectors are in fact Poisson.

3.1. Some Poisson structures of rank 2 in $\mathbb{R}^4$

Consider the Euclidean space $\mathbb{R}^4$, with coordinates (x_1, x_2, x_3, x_4). The equations of motion of the reduced non-holonomic systems encountered in the examples are observed to have integrals of motion which are related to the solutions of a system of differential equations of the type

$$\frac{\mathrm{d}x_3}{\mathrm{d}x_1} = h_3(x_1, x_3, x_4) \qquad \frac{\mathrm{d}x_4}{\mathrm{d}x_1} = h_4(x_1, x_3, x_4) \tag{4}$$

where h_3, h_4 are two given (smooth) functions of their arguments, which do not include x_2. We consider the system (4) as the Pfaffian system '$\theta_1 = 0, \theta_2 = 0$', where the 1-forms θ_1, θ_2 in $\mathbb{R}^4$ are given by

$$\theta_1 = -h_3(x_1, x_3, x_4)\,\mathrm{d}x_1 + \mathrm{d}x_3 \qquad \theta_2 = -h_4(x_1, x_3, x_4)\,\mathrm{d}x_1 + \mathrm{d}x_4. \tag{5}$$

These two 1-forms determine a codistribution integrable in the sense of Frobenius [26], since there exists a set of four 1-forms Δ_i^j such that $\mathrm{d}\theta_i = \Delta_i^j \wedge \theta_j$ for $i, j = 1, 2$. For example, we can take

$$\Delta_1^1 = \frac{\partial h_3}{\partial x_3}\,\mathrm{d}x_1 \qquad \Delta_1^2 = \frac{\partial h_3}{\partial x_4}\,\mathrm{d}x_1 \qquad \Delta_2^1 = \frac{\partial h_4}{\partial x_3}\,\mathrm{d}x_1 \qquad \Delta_2^2 = \frac{\partial h_4}{\partial x_4}\,\mathrm{d}x_1 \tag{6}$$

in order to satisfy the integrability condition. Thus, there will exist (locally) functions c_1, c_2 and Γ_i^j such that $\mathrm{d}c_i = \Gamma_i^j \theta_j$, $i, j = 1, 2$. The subvarieties solution of the Pfaffian system '$\theta_1 = 0, \theta_2 = 0$' are defined by the equations $c_i = b_i$, where b_i are constants, $i = 1, 2$.

More specifically, in the actual examples, the system (4) takes the form of a non-autonomous first-order system of linear differential equations

$$\frac{\mathrm{d}x_3}{\mathrm{d}x_1} = a_{11}(x_1)x_3 + a_{12}(x_1)x_4 \qquad \frac{\mathrm{d}x_4}{\mathrm{d}x_1} = a_{21}(x_1)x_3 + a_{22}(x_1)x_4$$

or, written in matrix form

$$\frac{\mathrm{d}}{\mathrm{d}x_1}\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = A(x_1)\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \tag{7}$$

where

$$A(x_1) = \begin{pmatrix} a_{11}(x_1) & a_{12}(x_1) \\ a_{21}(x_1) & a_{22}(x_1) \end{pmatrix}.$$

The previous functions c_i can be identified with the initial conditions of the solution of (7). In fact, such a solution can be expressed as $\mathbf{x} = g(x_1)\mathbf{c}$, where $\mathbf{x} = (x_3, x_4)^{\mathrm{T}}$, $\mathbf{c} = (c_1, c_2)^{\mathrm{T}}$, and $g(x_1)$ is a $GL(2, \mathbb{R})$-valued curve ($SL(2, \mathbb{R})$-valued curve if $\operatorname{tr} A(x_1) = 0$ for all x_1), solution of the right-invariant matrix system (see, e.g., [14, 15])

$$\frac{\mathrm{d}g}{\mathrm{d}x_1} g^{-1} = A(x_1). \tag{8}$$

Then, $\mathbf{c} = g^{-1}(x_1)\mathbf{x}$ gives the desired functions: with a slight abuse of notation, we have

$$\mathrm{d}\mathbf{c} = (\mathrm{d}g^{-1})\mathbf{x} + g^{-1}\,\mathrm{d}\mathbf{x} = -g^{-1}\,\mathrm{d}g g^{-1}\mathbf{x} + g^{-1}A\mathbf{x}\,\mathrm{d}x_1 = -g^{-1}\,\mathrm{d}g g^{-1}\mathbf{x} + g^{-1}\,\mathrm{d}g g^{-1}\mathbf{x} = 0$$

where we have used that $\mathrm{d}g^{-1} = -g^{-1}\,\mathrm{d}g g^{-1}$ and $A\,\mathrm{d}x_1 = \mathrm{d}g g^{-1}$. However, note that the solution of (8) cannot be expressed in an explicit way in the general case, and therefore, the functions c_1, c_2 cannot be explicitly written in general.

Now, we impose that the 1-forms (5) generate the kernel of the bivector in $\mathbb{R}^4$

$$\Lambda = \sum_{1\leqslant i<j\leqslant 4} \Lambda_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}. \tag{9}$$

The resulting bivectors will clearly have rank 2. Moreover, they are Poisson, according to the following result:

Theorem 1. *Consider in $\mathbb{R}^4$ a bivector of type (9), such that $\Lambda^\sharp(\theta_1) = 0$, $\Lambda^\sharp(\theta_2) = 0$, where θ_1, θ_2 are given by (5). Then the bivector is of the form*

$$\Lambda = -\Lambda_{12} U \wedge V$$

where

$$U = \frac{\partial}{\partial x_2} \qquad V = \frac{\partial}{\partial x_1} + h_3 \frac{\partial}{\partial x_3} + h_4 \frac{\partial}{\partial x_4} \tag{10}$$

and $\Lambda_{12} \in C^\infty(\mathbb{R}^4)$. Each of these bivectors is Poisson, and of rank 2 on points where $\Lambda_{12} \neq 0$.

Proof. The case of $\Lambda_{12} = 0$ is trivial. We will assume $\Lambda_{12} \neq 0$ in the domain of interest. Take Λ and θ_1, θ_2, as stated. The conditions $\Lambda^\sharp(\theta_1) = 0$, $\Lambda^\sharp(\theta_2) = 0$ give rise to an algebraic system for the six independent functions Λ_{ij}, which can easily be solved for five of them, in terms of the remaining one and the functions entering into the 1-forms. We choose Λ_{12} to be the undetermined function. Then the solution reads

$$\Lambda_{13} = \Lambda_{14} = \Lambda_{34} = 0 \qquad \Lambda_{23} = -\Lambda_{12} h_3 \qquad \Lambda_{24} = -\Lambda_{12} h_4$$

thus the resulting bivectors are as claimed. To see that each of them is Poisson, consider the bivector of the family with $\Lambda_{12} = -1$, i.e., $U \wedge V$. This bivector is of the form given in example 1, and $[U, V] = 0$, thus $U \wedge V$ is Poisson. It is moreover of rank 2, therefore by proposition 1, the claim follows. $\square$

Remark. Note that the vector fields U, V of the previous theorem satisfy $\theta_i(U) = \theta_i(V) = 0$, $i = 1, 2$, which in principle might seem a stronger condition than that the bivector (9) annihilates the 1-forms θ_1, θ_2.

Now, given a (Hamiltonian) function $H \in C^\infty(\mathbb{R}^4)$, the Hamiltonian vector field X_H with respect to a Poisson structure of the family described in theorem 1 takes the form

$$X_H = \Lambda^\sharp(\mathrm{d}H) = \Lambda_{12}\,[(VH)U - (UH)V] \tag{11}$$

where U and V are given by (10). Obviously, H is a first integral of X_H, since $X_H H = \Lambda(\mathrm{d}H, \mathrm{d}H) = 0$. Other two first integrals are the functions c_i such that $\mathrm{d}c_i = \Gamma_i^j \theta_j$, since by construction $X_H(c_i) = \Lambda(\mathrm{d}H, \mathrm{d}c_i) = \Gamma_i^j \Lambda(\mathrm{d}H, \theta_j) = 0, i = 1, 2$. These two first integrals are common to all Hamiltonian vector fields of type (11).

On the other hand, given a specific vector field X in $\mathbb{R}^4$, which is recognized to be of the form (11), it could be regarded as a Hamiltonian vector field with respect to *one specific* Poisson structure of the family described in theorem 1.

3.2. Some Poisson structures of rank 2 in $\mathbb{R}^5$

We will treat in this section analogous questions to that of the previous section, but now in the Euclidean space $\mathbb{R}^5$, with coordinates $(x_1, x_2, x_3, x_4, x_5)$.

The motivation is that typically, the reduced orbit spaces for the non-holonomic problems of interest, are semialgebraic varieties of $\mathbb{R}^5$, essentially determined by the zero-level set of a function $\phi \in C^\infty(\mathbb{R}^5)$, quadratic in its arguments, which are moreover subject to certain constraints. More specifically, in the examples it will have the form $\phi(x) = 0$, with $\phi(x) = x_2^2 + x_3^2 - (1 - x_1^2)x_5$, $|x_1| \leqslant 1$, and $x_5 \geqslant 0$, or with $\phi(x) = x_2^2 + x_3^2 - 4x_1x_5$, $x_1 \geqslant 0$, and $x_5 \geqslant 0$. However, for what follows ϕ can be in principle any differentiable function in $\mathbb{R}^5$.

We will consider then the Pfaffian system '$\theta_0 = 0, \theta_1 = 0, \theta_2 = 0$', where $\theta_0 = \mathrm{d}\phi$ and θ_1, θ_2 are 1-forms in $\mathbb{R}^5$ whose coordinate expression is again (5). These three 1-forms also determine a codistribution integrable in the sense of Frobenius in $\mathbb{R}^5$, because we have again $\mathrm{d}\theta_i = \Delta_i^j \wedge \theta_j$ with (6), $i, j = 1, 2$, and $\mathrm{d}\theta_0 = \mathrm{d}^2\phi = 0$.

We impose now that $\ker \Lambda^\sharp = \mathrm{span}\{\theta_0, \theta_1, \theta_2\}$, where Λ is the bivector in (some open set of) $\mathbb{R}^5$

$$\Lambda = \sum_{1 \leqslant i < j \leqslant 5} \Lambda_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}. \tag{12}$$

The resulting bivectors are again generically of rank 2 and Poisson, as follows:

Theorem 2. *Consider in* $\mathbb{R}^5$ *a bivector of type (12), such that* $\Lambda^\sharp(\theta_0) = 0$, $\Lambda^\sharp(\theta_1) = 0$ *and* $\Lambda^\sharp(\theta_2) = 0$, *where* $\theta_0 = \mathrm{d}\phi$, *and* θ_1, θ_2 *are given by (5). Then the bivector is of the form*

$$\Lambda = f[(Z\phi)U \wedge V + Y \wedge Z] \tag{13}$$

where

$$U = \frac{\partial}{\partial x_2} \qquad V = \frac{\partial}{\partial x_1} + h_3 \frac{\partial}{\partial x_3} + h_4 \frac{\partial}{\partial x_4} \qquad Z = \frac{\partial}{\partial x_5} \tag{14}$$

$$Y = (U\phi)V - (V\phi)U \tag{15}$$

and $f \in C^\infty(\mathbb{R}^5)$. *Each of these bivectors is Poisson, and of rank 2 on points where* $f \neq 0$.

Proof. Once more, the case of $f = 0$ is trivial, thus we will assume again that $f \neq 0$ in the domain of interest. Take $\Lambda, \theta_0, \theta_1$ and θ_2 as stated. The idea of the proof is similar to that of theorem 1. First of all, since the kernel of $\Lambda^\sharp$ has generically dimension 3, then the rank of $\Lambda^\sharp$ is 2. The conditions $\Lambda^\sharp(\theta_0) = 0$, $\Lambda^\sharp(\theta_1) = 0$ and $\Lambda^\sharp(\theta_2) = 0$ give rise again to an algebraic system for the functions Λ_{ij}, out of which all can be solved for except one of them, namely Λ_{12}, which we will write as $-(\partial\phi/\partial x_5)f$. The solution then reads

$$\Lambda_{13} = \Lambda_{14} = \Lambda_{34} = 0 \qquad \Lambda_{23} = f h_3 \frac{\partial \phi}{\partial x_5} \qquad \Lambda_{24} = f h_4 \frac{\partial \phi}{\partial x_5}$$

$$\Lambda_{15} = f \frac{\partial \phi}{\partial x_2} \qquad \Lambda_{35} = f h_3 \frac{\partial \phi}{\partial x_2}$$

$$\Lambda_{45} = f h_4 \frac{\partial \phi}{\partial x_2} \qquad \Lambda_{25} = -f \left(\frac{\partial \phi}{\partial x_1} + h_3 \frac{\partial \phi}{\partial x_3} + h_4 \frac{\partial \phi}{\partial x_4} \right)$$

thus the resulting bivectors take the stated form. To see that each of them is Poisson, consider the bivector of the family with $f = 1$, i.e., $\Lambda_0 = \overline{U} \wedge V + Y \wedge Z$, where $\overline{U} = (Z\phi)U$. We have to show that the Schouten–Nijenhuis bracket of Λ_0 with itself vanish, i.e., $[\Lambda_0, \Lambda_0] = 0$. By linearity and using the first property of (1) we have

$$[\Lambda_0, \Lambda_0] = [\overline{U} \wedge V, \overline{U} \wedge V] + 2[\overline{U} \wedge V, Y \wedge Z] + [Y \wedge Z, Y \wedge Z].$$

By example 1 we know that $[\overline{U} \wedge V, \overline{U} \wedge V] = 2\overline{U} \wedge V \wedge [\overline{U}, V]$ and analogously, $[Y \wedge Z, Y \wedge Z] = 2Y \wedge Z \wedge [Y, Z]$. Now, using again the second and third properties of (1) we can write

$$[\overline{U} \wedge V, Y \wedge Z] = V \wedge Z \wedge [\overline{U}, Y] - \overline{U} \wedge Z \wedge [V, Y] + Y \wedge V \wedge [\overline{U}, Z] - Y \wedge \overline{U} \wedge [V, Z].$$

We have to calculate now some Lie brackets. We have $[U, V] = [V, Z] = [U, Z] = 0$ but

$$[\overline{U}, V] = -[V(Z\phi)]U \qquad [Y, Z] = -[Z(U\phi)]V + [Z(V\phi)]U$$
$$[\overline{U}, Y] = (Z\phi)[U(U\phi)]V - \{(Z\phi)[U(V\phi)] + (U\phi)[V(Z\phi)] - (V\phi)[U(Z\phi)]\}U$$
$$[V, Y] = [V(U\phi)]V - [V(V\phi)]U \qquad [\overline{U}, Z] = -[Z(Z\phi)]U.$$

Then, summing up, we have

$$[\Lambda_0, \Lambda_0] = 2U \wedge V \wedge Z\{(Z\phi)([V, U]\phi) + (U\phi)([Z, V]\phi) + (V\phi)([U, Z]\phi)\} = 0.$$

Since the rank of any of the Λ, and in particular Λ_0, is 2, applying proposition 1 ends the proof. □

Remark. Note that the vector fields U, V, Y and Z of theorem 2 satisfy $\theta_i(U) = \theta_i(V) = \theta_i(Y) = \theta_i(Z) = 0, i = 1, 2, \theta_0(Y) = Y(\phi) = 0$ and $(U \wedge V)\phi - Y = 0$. These requirements might seem *a priori* to be stronger conditions to that imposed in the theorem.

If we are given now a (Hamiltonian) function $H \in C^\infty(\mathbb{R}^5)$, the Hamiltonian vector field X_H with respect to a Poisson structure of the family described in theorem 2 reads, using (15),

$$\begin{aligned} X_H = \Lambda^\sharp(\mathrm{d}H) = f\{&[(ZH)(V\phi) - (Z\phi)(VH)]U + [(Z\phi)(UH) - (ZH)(U\phi)]V \\ &+ [(U\phi)(VH) - (V\phi)(UH)]Z\} \end{aligned} \tag{16}$$

where U, V and Z are given by (14). By construction H is a first integral of X_H. Other first integrals are the functions c_i such that $\mathrm{d}c_i = \Gamma_i^j \theta_j$, as in the previous section. These two first integrals are common to all Hamiltonian vector fields of type (16).

However, given a specific vector field X_H of type (16), it fixes the specific function f and therefore the specific Poisson bivector of the family (13) with respect to which X_H is Hamiltonian.

4. Examples

In this section, we will show how the preceding results can be directly applied in the cases of reduced systems corresponding to specific examples of non-holonomic systems, i.e., the rolling disc, the Routh's sphere, the ball rolling on a surface of revolution and its special case of a ball rolling inside a cylinder.

4.1. The rolling disc

For this example we will follow the treatment and use some of the results of [18], see details therein. This problem has been treated as well, e.g., in [4, 7, 19, 33, 35]. Consider a homogeneous disc, which rolls without slipping on a horizontal plane under the influence of a vertical gravitational field of strenght g. The resulting non-holonomic system has two evident symmetry groups. One is the symmetry group $E(2)$ consisting of translations in the horizontal plane and rotations about the vertical axis, and the second is the S^1 symmetry consisting of rotations about the principal axis perpendicular to the plane of the disc.

After these two symmetries have been reduced out, in particular by using invariant theory for the reduction of the S^1 symmetry, it is obtained a system giving the evolution on the reduced orbit space, which is a semialgebraic variety of $\mathbb{R}^5$. In particular, the system can be restricted to a smooth open subset as it has been done in [18].

Thus, consider a reference homogeneous disc of radius r and mass m, lying flat in a fixed reference frame with centre of mass at the origin. The position of the moving disc is given by transforming the position of the reference disc by means of a translation a (e.g., of the centre of mass) and a rotation A. The tensor of inertia I with respect to the principal axes of the disc is diagonal, $I = \mathrm{diag}(I_1, I_1, I_3)$. Let us call e_3 the vertical unitary vector in the fixed frame of reference. We define the unitary vector u with respect to that frame as the pre-image of $-e_3$ under the rotation A, $u = -A^{-1}e_3$. The vector s in the fixed disc, rotated by A gives the vector in the moving disc pointing from the centre of mass to the point of contact of the moving disc with the horizontal plane. If we denote $\hat{u} = u - \langle u, e_3\rangle e_3$, the relation between s and u is $s = r\hat{u}/|\hat{u}|$. We denote by $(\omega_1, \omega_2, \omega_3)$ the components of the angular velocity vector ω of the disc.

Following [18], after the mentioned symmetry group $E(2)$ is reduced, the equations of motion read

$$\begin{aligned} \frac{\mathrm{d}(I\omega)}{\mathrm{d}t} &= I\omega\times\omega - mr^2\frac{\mathrm{d}\omega}{\mathrm{d}t} + m\left\langle\frac{\mathrm{d}\omega}{\mathrm{d}t}, s\right\rangle s + m\langle s,\omega\rangle\frac{\mathrm{d}s}{\mathrm{d}t} + m\langle\omega, s\rangle(\omega\times s) - mg(u\times s) \\ \frac{\mathrm{d}u}{\mathrm{d}t} &= u\times\omega \end{aligned} \tag{17}$$

which have a first integral given by the total energy of the disc

$$H = \tfrac{1}{2}\langle I\omega,\omega\rangle + \tfrac{1}{2}\langle\omega\times s, \omega\times s\rangle + mg\langle s, u\rangle. \tag{18}$$

The second of equations (17) expresses the non-holonomic constraint of rolling without slipping, i.e., instantaneous velocity of the point of contact equal to zero.

We recall briefly now how the further reduction of the S^1 symmetry is performed. Let us denote by (u_1, u_2, u_3) the components of u. The S^1 symmetry action consists of rotating both vectors u and ω simultaneously as mentioned, and it is not a free action since the isotropy subgroup of pairs $((0, 0, \pm 1), (0, 0, \omega_3))$ is S^1. Thus, we will use invariant theory in order to perform the reduction. A set of invariants for this action is easily constructed [18]:

$$\begin{aligned} &\sigma_1 = u_3 && \sigma_2 = u_2\omega_1 - u_1\omega_2 && \sigma_3 = u_1\omega_1 + u_2\omega_2 \\ &\sigma_4 = \omega_3 && \sigma_5 = \omega_1^2 + \omega_2^2 && \sigma_6 = u_1^2 + u_2^2 \end{aligned} \tag{19}$$

with the relations

$$\sigma_2^2 + \sigma_3^2 = \sigma_5\sigma_6 \qquad \sigma_5 \geqslant 0 \quad \sigma_6 \geqslant 0. \tag{20}$$

Since u is a unitary vector, we have that $\sigma_6 + \sigma_1^2 = 1$ and $|\sigma_1| \leqslant 1$, thus the completely reduced orbit space M is the semialgebraic variety of $\mathbb{R}^5$

$$M = \{(\sigma_1, \dots, \sigma_5) \in \mathbb{R}^5 \mid \phi(\sigma) = 0, |\sigma_1| \leqslant 1, \sigma_5 \geqslant 0\} \tag{21}$$

where $\phi \in C^\infty(\mathbb{R}^5)$ is the polynomial function $\phi(\sigma) = \sigma_2^2 + \sigma_3^2 - \big(1 - \sigma_1^2\big)\sigma_5$. However, M is *not* a smooth submanifold of $\mathbb{R}^5$. The singular points of M are

$$\Pi_\pm = \{(\pm 1, 0, 0, \sigma_4, \sigma_5) \in \mathbb{R}^5 \mid \sigma_4 \in \mathbb{R}, \sigma_5 \geqslant 0\}. \tag{22}$$

The non-smoothness of M is due to the fact that the S^1 action is not free, see [17].

The somehow redundant variables $(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5)$ therefore parametrize the reduced orbit space M. The induced system from (17) will be written in terms of the orbit variables:

simply calculating their time derivatives, using the equations of motion (17) and that $I_1 = \frac{1}{4}mr^2$ and $I_3 = \frac{1}{2}mr^2$, we arrive to the following system:

$$\begin{aligned}
\dot{\sigma}_1 &= \sigma_2 \\
\dot{\sigma}_2 &= \frac{6}{5}\sigma_3\sigma_4 - \sigma_1\sigma_5 + \frac{4}{5}\frac{\sigma_1\sigma_3^2}{1-\sigma_1^2} + \lambda\sigma_1\sqrt{1-\sigma_1^2} \\
\dot{\sigma}_3 &= -2\sigma_2\sigma_4 \\
\dot{\sigma}_4 &= -\frac{2}{3}\frac{1}{1-\sigma_1^2}\sigma_2\sigma_3 \\
\dot{\sigma}_5 &= 2\sigma_2\left(\frac{\lambda\sigma_1}{\sqrt{1-\sigma_1^2}} + \frac{4}{5}\frac{\sigma_1\sigma_3^2}{\left(1-\sigma_1^2\right)^2} - \frac{4}{5}\frac{\sigma_3\sigma_4}{1-\sigma_1^2}\right)
\end{aligned} \tag{23}$$

where $\lambda = \frac{4}{5}\frac{g}{r}$ and the dot means derivative with respect to time. The reduced energy, obtained from (18), reads

$$E = \frac{\sigma_5}{2} + \frac{3}{4}\sigma_4^2 - \frac{2}{5}\frac{\sigma_3^2}{1-\sigma_1^2} + \lambda\sqrt{1-\sigma_1^2}. \tag{24}$$

Although in principle expressions (23) and (24) are only defined on M, their right-hand sides make sense for $\mathcal{D} = \mathbb{R}^5\backslash(\{(\pm 1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) \mid \sigma_2\sigma_3 \neq 0\} \cup \{(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) \mid |\sigma_1| > 1\})$, so we will consider this extended domain for the vector field X whose integral curves are given by (23) and the reduced energy function E.

However, if we restrict ourselves to the original domain M, and moreover to points with $|\sigma_1| < 1$, we can define a smooth open dense subset $\overline{M} \subset M$ given by

$$\overline{M} = \left\{ (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) \in \mathbb{R}^5 \,\middle|\, \sigma_5 = \frac{\sigma_2^2 + \sigma_3^2}{1-\sigma_1^2}, |\sigma_1| < 1 \right\} \tag{25}$$

diffeomorphic to $\mathbb{R}^4$ [18]. The induced vector field $\overline{X}$ and energy $\overline{E}$ on $\overline{M}$ can be easily found from (23) and (24) by solving for σ_5. The integral curves of $\overline{X}$ are the solutions of the system

$$\begin{aligned}
\dot{\sigma}_1 &= \sigma_2 \\
\dot{\sigma}_2 &= \frac{6}{5}\sigma_3\sigma_4 - \frac{\sigma_1}{1-\sigma_1^2}\sigma_2^2 - \frac{1}{5}\frac{\sigma_1}{1-\sigma_1^2}\sigma_3^2 + \lambda\sigma_1\sqrt{1-\sigma_1^2} \\
\dot{\sigma}_3 &= -2\sigma_2\sigma_4 \\
\dot{\sigma}_4 &= -\frac{2}{3}\frac{1}{1-\sigma_1^2}\sigma_2\sigma_3
\end{aligned} \tag{26}$$

meanwhile

$$\overline{E} = \frac{1}{2}\frac{\sigma_2^2}{1-\sigma_1^2} + \frac{1}{10}\frac{\sigma_3^2}{1-\sigma_1^2} + \frac{3}{5}\sigma_4^2 + \lambda\sqrt{1-\sigma_1^2}. \tag{27}$$

These expressions are equations (18) and (19) of [18], respectively.

The reduced vector field X satisfies $X(E) = 0$ as well as $X(\phi) = 0$ in $\mathcal{D}$, meanwhile $\overline{X}(\overline{E}) = 0$ in $\overline{M}$. In addition, X has a family of equilibrium points belonging to the singular set $\Pi_\pm$, called *singular equilibria*, given by $\{(\pm 1, 0, 0, \sigma_4, 0) \mid \sigma_4 \in \mathbb{R}\}$, and a family of *regular equilibria* given by the set of constants

$$\left\{ (\sigma_{10}, 0, \sigma_{30}, \sigma_{40}, \sigma_{50}) \in \mathcal{D} \,\middle|\, \frac{6}{5}\sigma_{30}\sigma_{40} - \sigma_{10}\sigma_{50} + \frac{4}{5}\frac{\sigma_{10}\sigma_{30}^2}{1-\sigma_{10}^2} + \lambda\sigma_{10}\sqrt{1-\sigma_{10}^2} = 0 \right\}.$$

These regular equilibria, in the original system, correspond to periodic motions of the disc in which the point of contact describes a circle and the centre of mass stands at a constant height. These motions are contained in the set of *steady motions* of the rolling disc, according to Routh's terminology [33, 35]. They have received an extensive treatment in [18], although by using the system (26).

Now, both of the systems (23) and (26) admit two first integrals related to the solutions (in the sense explained in section 3.1) of the non-autonomous linear system

$$\frac{\mathrm{d}\sigma_3}{\mathrm{d}\sigma_1} = -2\sigma_4 \qquad \frac{\mathrm{d}\sigma_4}{\mathrm{d}\sigma_1} = -\frac{2}{3}\frac{\sigma_3}{1-\sigma_1^2} \tag{28}$$

which can be written in matrix form as

$$\frac{\mathrm{d}}{\mathrm{d}\sigma_1}\begin{pmatrix}\sigma_3\\ \sigma_4\end{pmatrix} = \begin{pmatrix}0 & -2\\ -\frac{2}{3}\frac{1}{1-\sigma_1^2} & 0\end{pmatrix}\begin{pmatrix}\sigma_3\\ \sigma_4\end{pmatrix}. \tag{29}$$

This equation is the same as equation (69) of [18], where its solutions have been studied in great detail, including their asymptotic behaviour.

However, the important point for us is that the systems (23) and (26) are good candidates to be formulated as Hamiltonian systems with respect to Poisson structures of the type described in theorems 2 and 1, respectively. Let $\theta_0 = \mathrm{d}\phi$ and θ_1, θ_2 be the 1-forms, defined in $\overline{M}$ (resp. $\mathcal{D}$) by

$$\theta_1 = 2\sigma_4\,\mathrm{d}\sigma_1 + \mathrm{d}\sigma_3 \qquad \theta_2 = \frac{2}{3}\frac{\sigma_3}{1-\sigma_1^2}\,\mathrm{d}\sigma_1 + \mathrm{d}\sigma_4.$$

Applying the results of sections 3.1 and 3.2 to these 1-forms, we have

Proposition 2. *The bivectors of the form* $\overline{\Lambda} = -\Lambda_{12} U \wedge V$*, defined in* $\overline{M}$*, where*

$$U = \frac{\partial}{\partial\sigma_2} \qquad V = \frac{\partial}{\partial\sigma_1} - 2\sigma_4\frac{\partial}{\partial\sigma_3} - \frac{2}{3}\frac{\sigma_3}{1-\sigma_1^2}\frac{\partial}{\partial\sigma_4} \tag{30}$$

and $\Lambda_{12} \in C^\infty(\overline{M})$ *is a non-vanishing function, are Poisson tensors of rank 2 in* $\overline{M}$*.*

The vector field $\overline{X}$ *in* $\overline{M}$*, whose integral curves are the solutions of (26), is a Hamiltonian vector field with respect to the Poisson bivector* $\overline{\Lambda}$ *with the specific function* $\Lambda_{12} = 1 - \sigma_1^2$ *and Hamiltonian function* $\overline{E}$ *given by (27), i.e.,* $\overline{X} = \overline{\Lambda}^\sharp(\mathrm{d}\overline{E})$ *in* $\overline{M}$*.*

Proposition 3. *The bivectors* $\Lambda = f[(Z\phi)U \wedge V + Y \wedge Z]$*, defined in* $\mathcal{D} \subset \mathbb{R}^5$*, where* U *and* V *are given by (30),* $Z = \partial/\partial\sigma_5$*,* $Y = (U\phi)V - (V\phi)U$*, and* $f \in C^\infty(\mathcal{D})$ *is a non-vanishing function, are Poisson tensors of rank 2 in* $\mathcal{D}$*, except in the set of singular equilibria, where they vanish.*

The vector field X *in* $\mathcal{D}$*, whose integral curves are the solutions of (23), is a Hamiltonian vector field with respect to the Poisson bivector* Λ *with the specific function* $f = 1$ *and Hamiltonian function* E *given by (24), i.e.,* $X = \Lambda^\sharp(\mathrm{d}E)$ *in* $\mathcal{D}$*.*

Both propositions can be proved by direct computations.

The Poisson Hamiltonian structure of the systems (23) and (26) could be used to have an interpretation of their geometry. For example, the invariant submanifolds mentioned in the analysis of the reduced vector field (26) in [18], could be understood as the symplectic leaves of the rank-two Poisson structure(s) $\overline{\Lambda}$ of proposition 2.

4.2. Routh's sphere

For this example, we will follow the treatment and use some of the results of [17], see details therein. This problem has been treated as well, e.g., in [4, 7, 20, 33, 35]. Consider a sphere of mass m and radius r with its centre of mass at a distance α $(0 < \alpha < r)$ from its geometric centre. The line joining both centres is a principal axis of inertia, with associated moment of inertia I_3. Any axis orthogonal to the previous, passing through the geometric centre, has an associated moment of inertia I_1. This sphere is supposed to roll on a horizontal plane under the influence of a vertical gravitational field of strenght g. The resulting non-holonomic system has as well two symmetry groups. One is again the group $E(2)$ consisting of translations in the horizontal plane and rotations about the vertical axis. The other is the S^1 symmetry consisting of rotations about the principal axis of inertia which joins the centre of mass and the geometric centre of the ball.

Again, after these symmetries have been reduced out by a similar procedure to that of the rolling disc, it is obtained a system giving the evolution on the reduced orbit space, which coincides with that of the rolling disc.

Therefore, let us consider a reference ball as the one described, with the geometric centre at the origin, and the centre of mass at the point $-\alpha e_3$, where e_3 denotes the vertical unitary vector in this fixed frame. The position and attitude of the moving ball is given by transforming the position of the reference ball by means of a translation a (e.g., of the centre of mass) and a rotation A. We denote by s the vector in the fixed sphere such that rotated by A gives the vector in the moving sphere pointing from the centre of mass to the point of contact. The unitary vector u in the fixed frame is the pre-image of $-e_3$ under the rotation A, $u = -A^{-1}e_3$. The relation between u and s is $a_3 = \langle s, u\rangle$. The components of the angular velocity ω of the ball will be denoted by $(\omega_1, \omega_2, \omega_3)$.

Following [17], after the reduction of the mentioned $E(2)$ symmetry, the equations of motion read

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(I\omega + ms \times (\omega \times s)) = I\omega \times \omega + m\frac{\mathrm{d}s}{\mathrm{d}t} \times (\omega \times s) + m\langle \omega, s\rangle(\omega \times s) + mg(u \times s)\\
&\frac{\mathrm{d}u}{\mathrm{d}t} = u \times \omega
\end{aligned} \tag{31}$$

which have a first integral given by the total energy of the ball

$$H = \tfrac{1}{2}\langle I\omega, \omega\rangle + \tfrac{1}{2}\langle \omega \times s, \omega \times s\rangle + mg\langle s, u\rangle. \tag{32}$$

The second of equations (31) expresses again the non-holonomic constraint of rolling without slipping.

Now, the reduction of the S^1 symmetry is performed in an analogous way as in the case of the rolling disc, see section 4.1, where (u_1, u_2, u_3) denote as well the components of u. The S^1 action consists of rotating both vectors u, ω simultaneously, with respect to the principal axis joining the geometric and mass centres. This action is not free, since S^1 leaves invariant pairs of vectors of the form $((0, 0, \pm 1), (0, 0, \omega_3))$. The corresponding set of invariants is again (19) with the relations (20). Thus, the reduced orbit space M is the semialgebraic variety of $\mathbb{R}^5$ described in the previous example of the rolling disc, with the same notations.

However, the reduced system reads now, using (31),

$$\begin{aligned}
\dot\sigma_1 &= \sigma_2\\
T(\sigma_1)\dot\sigma_2 &= (I_3 + mr^2 + mr\alpha\sigma_1)\sigma_3\sigma_4 - mg\alpha\big(1 - \sigma_1^2\big)\\
&\qquad - \sigma_5\big(mr\alpha + (I_1 + m\alpha^2 + mr^2)\sigma_1 + mr\alpha\sigma_1^2\big)\\
\dot\sigma_3 &= -I_3\frac{\sigma_2\sigma_4}{P(\sigma_1)}(I_3 + mr^2 + mr\alpha\sigma_1)\\
\dot\sigma_4 &= -mr\frac{\sigma_2\sigma_4}{P(\sigma_1)}(I_3\alpha + r(I_3 - I_1)\sigma_1)\\
T(\sigma_1)\dot\sigma_5 &= -2mr\alpha\sigma_2\sigma_5 - 2mg\alpha\sigma_2 - 2mr^2(I_3 - I_1)\frac{I_3 + mr^2 + mr\alpha\sigma_1}{P(\sigma_1)}\sigma_2\sigma_3\sigma_4
\end{aligned} \tag{33}$$

where $P(\sigma_1) = I_1 I_3 + mr^2 I_1\left(1-\sigma_1^2\right) + mI_3(\alpha + r\sigma_1)^2$ and $T(\sigma_1) = I_1 + mr^2 + m\alpha^2 + 2mr\alpha\sigma_1$. The reduced energy is

$$E = \tfrac{1}{2}(T(\sigma_1)\sigma_5 + (I_3 + mr^2)\sigma_4^2 - mr^2(\sigma_3 + \sigma_1\sigma_4)^2) + m\alpha(g\sigma_1 - r\sigma_3\sigma_4). \tag{34}$$

These are equations (23), (24) and (25) in [17]. In this case expressions (33) and (34) make sense for all $\mathcal{D} = \mathbb{R}^5$. We will consider this extended domain for the vector field X whose integral curves are given by (33) and for the reduced energy function E.

Restricting ourselves to points in M with $|\sigma_1| < 1$, we find again the smooth submanifold $\overline{M} \subset M$ given by (25). The integral curves of the projected vector field $\overline{X}$ are the solutions of the system ([17], equation (38))

$$\begin{aligned}
\dot\sigma_1 &= \sigma_2 \\
T(\sigma_1)\dot\sigma_2 &= (I_3 + mr^2 + mr\alpha\sigma_1)\sigma_3\sigma_4 - mg\alpha\left(1-\sigma_1^2\right) \\
&\quad - \frac{\sigma_2^2 + \sigma_3^2}{1-\sigma_1^2}\left(mr\alpha + (I_1 + m\alpha^2 + mr^2)\sigma_1 + mr\alpha\sigma_1^2\right) \\
\dot\sigma_3 &= -I_3\frac{\sigma_2\sigma_4}{P(\sigma_1)}(I_3 + mr^2 + mr\alpha\sigma_1) \\
\dot\sigma_4 &= -mr\frac{\sigma_2\sigma_4}{P(\sigma_1)}(I_3\alpha + r(I_3 - I_1)\sigma_1)
\end{aligned} \tag{35}$$

and the restricted reduced energy $\overline{E}$ is

$$\overline{E} = \frac{1}{2}\left(T(\sigma_1)\frac{\sigma_2^2+\sigma_3^2}{1-\sigma_1^2} + (I_3 + mr^2)\sigma_4^2 - mr^2(\sigma_3 + \sigma_1\sigma_4)^2\right) + m\alpha(g\sigma_1 - r\sigma_3\sigma_4). \tag{36}$$

The reduced vector field X satisfies $X(E) = 0$ and $X(\phi) = 0$ in $\mathcal{D}$, and $\overline{X}(\overline{E}) = 0$ in $\overline{M}$. Moreover, X has a family of singular equilibrium points, belonging to the singular set $\Pi_\pm$, given by $\{(\pm 1, 0, 0, \sigma_4, 0) \mid \sigma_4 \in \mathbb{R}\}$, which physically correspond to the spinning of the ball about its symmetry axis when it is vertical (then the reduced energy becomes $\frac{1}{2}I_3\sigma_4^2 \pm mg\alpha$). It has as well a family of regular equilibria given by the set of constants

$$\{(\sigma_{10}, 0, \sigma_{30}, \sigma_{40}, \sigma_{50}) \in \mathbb{R}^5 \mid b(\sigma_{10}, \sigma_{30}, \sigma_{40}, \sigma_{50}) = 0\}$$

where $b(\sigma_1, \sigma_3, \sigma_4, \sigma_5) = (I_3 + mr^2 + mr\alpha\sigma_1)\sigma_3\sigma_4 - mg\alpha\left(1-\sigma_1^2\right) - \sigma_5\big(mr\alpha + (I_1 + m\alpha^2 + mr^2)\sigma_1 + mr\alpha\sigma_1^2\big)$. These regular equilibria, in the original system, correspond to periodic motions of the ball in which the point of contact describes a circle and the centre of mass stands at constant height.

In this case, both of the systems (33) and (35) admit two first integrals related to the solutions (in the sense of section 3.1) of the non-autonomous linear system

$$\frac{\mathrm{d}\sigma_3}{\mathrm{d}\sigma_1} = -\frac{I_3(I_3 + mr(r + \alpha\sigma_1))\sigma_4}{P(\sigma_1)} \qquad \frac{\mathrm{d}\sigma_4}{\mathrm{d}\sigma_1} = \frac{mr(I_1 r\sigma_1 - I_3(\alpha + r\sigma_1))\sigma_4}{P(\sigma_1)}. \tag{37}$$

Thus, the mentioned systems are other good candidates on which to apply the Poisson approach of section 3. Let $\theta_0 = \mathrm{d}\phi$ and θ_1, θ_2 be the 1-forms, defined in $\overline{M}$ (resp. $\mathcal{D}$) by

$$\begin{aligned}
\theta_1 &= \frac{I_3(I_3 + mr(r + \alpha\sigma_1))\sigma_4}{P(\sigma_1)}\,\mathrm{d}\sigma_1 + \mathrm{d}\sigma_3 \\
\theta_2 &= -\frac{mr(I_1 r\sigma_1 - I_3(\alpha + r\sigma_1))\sigma_4}{P(\sigma_1)}\,\mathrm{d}\sigma_1 + \mathrm{d}\sigma_4.
\end{aligned}$$

We have the following results:

Proposition 4. *The bivectors of the form* $\overline{\Lambda} = -\Lambda_{12} U \wedge V$, *defined in* $\overline{M}$, *where* $U = \partial/\partial\sigma_2$,

$$V = \frac{\partial}{\partial\sigma_1} - \frac{I_3(I_3 + mr(r + \alpha\sigma_1))\sigma_4}{P(\sigma_1)} \frac{\partial}{\partial\sigma_3} + \frac{mr(I_1 r\sigma_1 - I_3(\alpha + r\sigma_1))\sigma_4}{P(\sigma_1)} \frac{\partial}{\partial\sigma_4} \tag{38}$$

and $\Lambda_{12} \in C^\infty(\overline{M})$ *is a non-vanishing function, are Poisson tensors of rank 2 in* $\overline{M}$.

The vector field $\overline{X}$ *in* $\overline{M}$, *whose integral curves are the solutions of (35), is a Hamiltonian vector field with respect to the Poisson bivector* $\overline{\Lambda}$ *with the specific function* $\Lambda_{12} = \left(1 - \sigma_1^2\right)/T(\sigma_1)$ *and Hamiltonian function* $\overline{E}$ *given by (36), i.e.,* $\overline{X} = \overline{\Lambda}^\sharp(\mathrm{d}\overline{E})$ *in* $\overline{M}$.

Proposition 5. *The bivectors* $\Lambda = f[(Z\phi)U \wedge V + Y \wedge Z]$, *defined in* $\mathcal{D} = \mathbb{R}^5$, *where* U *and* V *are given as in proposition 4,* $Z = \partial/\partial\sigma_5$, $Y = (U\phi)V - (V\phi)U$, *and* $f \in C^\infty(\mathcal{D})$ *is a non-vanishing function, are Poisson tensors of rank 2 in* $\mathcal{D}$, *except in the set of singular equilibria, where they vanish.*

The vector field X *in* $\mathcal{D}$, *whose integral curves are the solutions of (33), is a Hamiltonian vector field with respect to the Poisson bivector* Λ *with the specific function* $f = 1/T(\sigma_1)$ *and Hamiltonian function* E *given by (34), i.e.,* $X = \Lambda^\sharp(\mathrm{d}E)$ *in* $\mathcal{D}$.

Both propositions can be proved as well by direct computations.

In this case, equations (37) can be explicitly integrated in an easy way. From the second of these equations we have the relation $\sigma_4\sqrt{P(\sigma_1)} = k$. Substituting into the first, we can also integrate to obtain the relation $I_1 r\sigma_3 + I_3(\alpha + r\sigma_1)\sigma_4 = j$. The constants k, j are integration constants (essentially the initial conditions of the system (37)). These two expressions are the desired first integrals (Casimir functions of the preceding Poisson structures). The second of them is known as *Jellet's integral*, see [17, 20] and references therein, see also p 184 of [7].

The invariant submanifolds thoroughly studied in [17], could be interpreted in this framework as the symplectic leaves of the rank-2 Poisson structure(s) $\overline{\Lambda}$ of proposition 4, determined by the level sets of the first integrals j and k.

4.3. Ball rolling on a surface of revolution

For this example we will follow the treatment and use some results of [25], see therein for more details. This problem has been treated as well, e.g., in [8, 33, 35]. In particular Routh, in the last of these references, noted the existence of two integrals of motion given by a system of two linear differential equations, solved them in special cases, and described a family of stationary periodic motions together with a necessary condition for their stability. Later, in [42], it has been shown that the condition is also sufficient. Both of [25] and [42] prove that the corresponding reduced system has integral curves consisting of either periodic orbits or equilibrium points.

Consider a homogeneous ball of mass m, radius r and moment of inertia M with respect to any principal axis. The ball rolls without slipping on a surface of revolution, under the influence of a vertical gravitational field of strength g. We take the origin of coordinates at a point of the axis of symmetry of the surface (the intersection of this axis with the surface at its vertex), and we consider a horizontal plane passing through it. We parametrize the position of the centre of mass of the ball by its coordinates (a_1, a_2) on this horizontal plane, and its height will be parametrized via the smooth *profile function* $\varphi : \mathbb{R} \to \mathbb{R}$ of the surface, $a_3 = \varphi\left(\sqrt{a_1^2 + a_2^2}\right)$. Note that not all surfaces of revolution can be parametrized well in this way, e.g., the cylinder, which requires a separate treatment, see section 4.4 below. We will assume that φ is a smooth even function, thus we will have that $\varphi^{(2k+1)}(0) = 0$, $k = 0, 1, 2, \ldots$.

We denote by $(\omega_1, \omega_2, \omega_3)$ the components of the angular velocity vector ω of the ball, and $(\gamma_1, \gamma_2, \gamma_3)$ the components of a unit vector γ normal to the surface at the point of contact (directed towards the centre of the ball). The unit vector in the vertical coordinate axis is e_3.

The equations of motion can be easily computed by the classical equations of the variation of the angular momentum, and implementing the non-holonomic constraint of non-slipping of the point of contact, i.e., that its instantaneous velocity vanishes. They read (with respect to the centre of mass of the ball, compare with equations (5), (7) of [25] and section 2 of [8])

$$\begin{aligned} &M\frac{\mathrm{d}\omega}{\mathrm{d}t} - mr^2\left(\frac{\mathrm{d}}{\mathrm{d}t}(\omega\times\gamma)\right)\times\gamma - mgre_3\times\gamma = 0 \\ &\dot{a} - r(\omega\times\gamma) = 0. \end{aligned} \tag{39}$$

The total mechanical energy of this system is then

$$H = \tfrac{1}{2}((M+mr^2)(\omega\cdot\omega) - mr^2(\gamma\cdot\omega)^2) + mga_3 \tag{40}$$

and is a first integral for the system (39).

The system (39) and the energy (40) admit a further reduction of the S^1 symmetry consisting of rotations of the system about the vertical axis, and thus rotating both of ω and γ simultaneously. This action, as in the previous cases, is not free, since the isotropy subgroup of pairs $((0, 0, 1), (0, 0, \omega_3))$ is S^1 (these pairs correspond to motions of the ball spinning around the vertical axis when being at the vertex of the surface), and we will use again invariant theory in order to perform the reduction, but now as it has been done in [25]. First of all, we define the vector v and the scalar w as follows: $v = r(\omega\times\gamma)$, $w = -r(\omega\cdot\gamma)$. Then, a full set of invariant polynomials, which parametrize the orbit space of the S^1 action, is

$$\begin{aligned} &p_1 = \tfrac{1}{2}\left(a_1^2 + a_2^2\right) \qquad p_2 = a_1v_1 + a_2v_2 \qquad p_3 = a_1v_2 - a_2v_1 \\ &p_4 = w \qquad p_5 = \tfrac{1}{2}\left(v_1^2 + v_2^2\right) \end{aligned} \tag{41}$$

with the relations

$$p_2^2 + p_3^2 - 4p_1p_5 = 0 \qquad p_1 \geqslant 0 \qquad p_5 \geqslant 0. \tag{42}$$

Therefore, the completely reduced orbit space P is now the semialgebraic variety of $\mathbb{R}^5$

$$P = \{(p_1, \ldots, p_5) \in \mathbb{R}^5 \mid \phi(p) = 0,\ p_1 \geqslant 0,\ p_5 \geqslant 0\} \tag{43}$$

where now $\phi \in C^\infty(\mathbb{R}^5)$ is the polynomial function $\phi(p) = p_2^2 + p_3^2 - 4p_1p_5$. P is *not* a smooth submanifold of $\mathbb{R}^5$, because the previous S^1 action is not free. Instead, P is homeomorphic to a cone in $\mathbb{R}^4$ times $\mathbb{R}$ [25], which can easily be seen from the relation $\phi(p) = 0$ when it is written as $p_2^2 + p_3^2 + (p_1 - p_5)^2 = (p_1 + p_5)^2$. The vertex of the cone is determined by $p_2 = p_3 = p_1 = p_5 = 0$, therefore the singular points of P are

$$\Pi = \{(0, 0, 0, p_4, 0) \in P \mid p_4 \in \mathbb{R}\}.$$

Calculating the time derivatives of the invariants, using (39), and the relations

$$\gamma_1 = -\frac{a_1}{\sqrt{2p_1}}\frac{\varphi'}{\sqrt{1+\varphi'^2}} \qquad \gamma_2 = -\frac{a_2}{\sqrt{2p_1}}\frac{\varphi'}{\sqrt{1+\varphi'^2}} \qquad \gamma_3 = \frac{1}{\sqrt{1+\varphi'^2}}$$

(we will use the notation $\varphi = \varphi(\sqrt{2p_1})$, $\varphi' = \varphi'(\sqrt{2p_1})$ and $\varphi'' = \varphi''(\sqrt{2p_1})$ in what follows) we arrive to the system in the reduced orbit space P

$$\begin{aligned} \dot{p}_1 &= p_2 \\ \dot{p}_2 &= \frac{1}{1+\varphi'}\left\{-\frac{M}{\alpha r^2}p_3p_4\frac{\varphi'}{\sqrt{2p_1}} - \frac{mg}{\alpha}\sqrt{2p_1}\varphi' + 2p_5 - p_2^2\frac{\varphi'}{\sqrt{2p_1}}\left(\varphi'' - \frac{\varphi'}{\sqrt{2p_1}}\right)\right\} \end{aligned}$$

$$\dot{p}_3 = \frac{M}{\alpha r^2} p_2 p_4 \frac{\varphi''}{1+\varphi'^2}$$
$$\dot{p}_4 = -\frac{p_2 p_3}{2p_1}\left(\frac{\varphi''}{1+\varphi'^2} - \frac{\varphi'}{\sqrt{2p_1}}\right)$$
$$\dot{p}_5 = \frac{p_2}{1+\varphi'}\left\{\frac{1}{2p_1}\left(\frac{M}{\alpha r^2}p_3p_4 - p_2^2\frac{\varphi'}{\sqrt{2p_1}}\right)\left(\varphi'' - \frac{\varphi'}{\sqrt{2p_1}}\right) - \frac{mg}{\alpha}\frac{\varphi'}{\sqrt{2p_1}} - 2p_5\frac{\varphi'^2}{2p_1}\right\} \tag{44}$$

and the reduced energy

$$E = \frac{M}{2r^2}p_4^2 + \alpha p_5 + \frac{\alpha\varphi'^2}{4p_1}p_2^2 + mg\varphi \tag{45}$$

where $\alpha = \frac{M+mr^2}{r^2}$. These are the equations found in lemmas 2.2 and 2.3 (i) of [25].

We observe that the right-hand sides of (44) and (45) make sense in an open set $\mathcal{D}$ of $\mathbb{R}^5$ larger than P, namely $\mathcal{D} = \mathbb{R}^5\backslash\{(p_1, p_2, p_3, p_4, p_5) \mid p_1 < 0\}$. This is due to the fact that they are defined in the limit $p_1 \to 0^+$, because of the assumption that the odd-order derivatives at 0 of φ vanish. (For points strictly in P with $p_1 = 0$ this assumption would not be necessary, since these points also have $p_2 = 0$, $p_3 = 0$.) We will consider the enlarged domain $\mathcal{D}$ for the vector field X whose integral curves are the solutions of (44), and also for the reduced energy (45), compare with p 500 of [25].

The regular stratum of P, i.e., $P\backslash\Pi$, can be covered by two charts [23], whose corresponding neighbourhoods can be chosen to be the smooth open dense subsets $\overline{P}_1, \overline{P}_2 \subset P$ given by

$$\begin{aligned}\overline{P}_1 &= \left\{(p_1, p_2, p_3, p_4, p_5) \in \mathbb{R}^5 \,\middle|\, p_5 = \frac{p_2^2 + p_3^2}{4p_1},\ p_1 > 0\right\}\\ \overline{P}_2 &= \left\{(p_1, p_2, p_3, p_4, p_5) \in \mathbb{R}^5 \,\middle|\, p_1 = \frac{p_2^2 + p_3^2}{4p_5},\ p_5 > 0\right\}.\end{aligned} \tag{46}$$

However, for our purposes here, it will be enough to consider just $\overline{P}_1$, in order to endow it with Poisson structures of the type described in theorem 1, which afterwards could be compared with the Poisson structure given originally in [8]. The procedure for $\overline{P}_2$ is analogous. Thus, we will consider the induced vector field $\overline{X}$ and energy $\overline{E}$ on $\overline{P}_1$, which can be found from (44) and (45) by solving for p_5. The integral curves of $\overline{X}$ are the solutions of the system

$$\begin{aligned}\dot{p}_1 &= p_2\\ \dot{p}_2 &= \frac{1}{1+\varphi'}\left\{-\frac{M}{\alpha r^2}p_3p_4\frac{\varphi'}{\sqrt{2p_1}} - \frac{mg}{\alpha}\sqrt{2p_1}\varphi' + \frac{p_2^2+p_3^2}{2p_1} - p_2^2\frac{\varphi'}{\sqrt{2p_1}}\left(\varphi'' - \frac{\varphi'}{\sqrt{2p_1}}\right)\right\}\\ \dot{p}_3 &= \frac{M}{\alpha r^2}p_2p_4\frac{\varphi''}{1+\varphi'^2}\\ \dot{p}_4 &= -\frac{p_2p_3}{2p_1}\left(\frac{\varphi''}{1+\varphi'^2} - \frac{\varphi'}{\sqrt{2p_1}}\right)\end{aligned} \tag{47}$$

meanwhile

$$\overline{E} = \frac{M}{2r^2}p_4^2 + \alpha\frac{p_2^2+p_3^2}{4p_1} + \frac{\alpha\varphi'^2}{4p_1}p_2^2 + mg\varphi. \tag{48}$$

Now, the reduced vector field X satisfies $X(E) = 0$ and $X(\phi) = 0$ in $\mathcal{D}$, and $\overline{X}(\overline{E}) = 0$ in $\overline{P}_1$. The vector field X has a family of singular equilibrium points consisting of the singular set Π, that is, $\{(0, 0, 0, p_4, 0) \mid p_4 \in \mathbb{R}\}$, which as already mentioned, correspond to the spinning of the ball about the vertical when being at the vertex of the surface (then the reduced energy becomes $\frac{M}{2r^2}p_4^2 + mg\varphi(0)$). X has as well a family of regular equilibria given by the set of constants

$$\left\{(p_{10}, 0, p_{30}, p_{40}, p_{50}) \in \mathcal{D} \;\middle|\; 2p_{50} - \frac{mg}{\alpha}\sqrt{2p_{10}}\varphi'\big(\sqrt{2p_{10}}\big) - \frac{M}{\alpha r^2}\frac{\varphi'\big(\sqrt{2p_{10}}\big)}{\sqrt{2p_{10}}}p_{30}p_{40} = 0\right\}.$$

These regular equilibria correspond in the original system to rotations of the ball along a parallel of the surface of revolution at constant height.

In addition, both of the systems (44) and (47) admit two first integrals of motion related to the solutions (in the sense explained in section 3.1) of the non-autonomous linear system (see also [25], lemma 2.3. (ii))

$$\frac{\mathrm{d}p_3}{\mathrm{d}p_1} = \frac{M}{\alpha r^2}p_4\frac{\varphi''}{1+\varphi'^2} \qquad \frac{\mathrm{d}p_4}{\mathrm{d}p_1} = -\frac{p_3}{2p_1}\left(\frac{\varphi''}{1+\varphi'^2} - \frac{\varphi'}{\sqrt{2p_1}}\right). \tag{49}$$

Let $\theta_0 = \mathrm{d}\phi$ and θ_1, θ_2, be the 1-forms, defined in $\overline{P}_1$ (resp. $\mathcal{D}$) by

$$\theta_1 = \frac{M}{\alpha r^2}p_4\frac{\varphi''}{1+\varphi'^2}\,\mathrm{d}p_1 - \mathrm{d}p_3 \qquad \theta_2 = \frac{p_3}{2p_1}\left(\frac{\varphi''}{1+\varphi'^2} - \frac{\varphi'}{\sqrt{2p_1}}\right)\mathrm{d}p_1 + \mathrm{d}p_4.$$

We have the following results, applying the theorems of section 3, which can be proved by direct computations:

Proposition 6. *The bivectors of the form $\overline{\Lambda} = -\Lambda_{12}U \wedge V$, defined in $\overline{P}_1$, where*

$$U = \frac{\partial}{\partial p_2} \qquad V = \frac{\partial}{\partial p_1} + \frac{M}{\alpha r^2}p_4\frac{\varphi''}{1+\varphi'^2}\frac{\partial}{\partial p_3} - \frac{p_3}{2p_1}\left(\frac{\varphi''}{1+\varphi'^2} - \frac{\varphi'}{\sqrt{2p_1}}\right)\frac{\partial}{\partial p_4} \tag{50}$$

and $\Lambda_{12} \in C^\infty(\overline{P}_1)$ is a non-vanishing function, are Poisson tensors of rank 2 in $\overline{P}_1$.

The vector field $\overline{X}$ in $\overline{P}_1$, whose integral curves are the solutions of (47), is a Hamiltonian vector field with respect to the Poisson bivector $\overline{\Lambda}$ with the specific function $\Lambda_{12} = 2p_1/\alpha(1+\varphi'^2)$ and Hamiltonian function $\overline{E}$ given by (48), i.e., $\overline{X} = \overline{\Lambda}^\sharp(\mathrm{d}\overline{E})$ in $\overline{P}_1$.

Proposition 7. *The bivectors $\Lambda = f[(Z\phi)U \wedge V + Y \wedge Z]$, defined in $\mathcal{D} \subset \mathbb{R}^5$, where U and V are given by (50), $Z = \partial/\partial p_5$, $Y = (U\phi)V - (V\phi)U$, and $f \in C^\infty(\mathcal{D})$ is a non-vanishing function, are Poisson tensors of rank 2 in $\mathcal{D}$, except in the set of singular equilibria, where they vanish.*

The vector field X in $\mathcal{D}$, whose integral curves are the solutions of (44), is a Hamiltonian vector field with respect to the Poisson bivector Λ with the specific function $f = 1/2\alpha(1+\varphi'^2)$ and Hamiltonian function E given by (45), i.e., $X = \Lambda^\sharp(\mathrm{d}E)$ in $\mathcal{D}$.

Remark. For the present case, a Poisson structure analogous to one of the structures $\overline{\Lambda}$ of proposition 6 has been found, to the best of our knowledge, by the first time in [8], see their equation (3.11) for $\lambda = 0$. In fact, up to a rescaling, they are the same, by using the identifications

$$x_2 = \frac{M}{r}p_4\sqrt{2p_1}\frac{\sqrt{1+\varphi'^2}}{\varphi'} \qquad x_3 = -\frac{M}{r}\frac{\sqrt{2p_1}p_4 + p_3\varphi'}{\sqrt{2p_1}\sqrt{1+\varphi'^2}} \qquad x_4 = \alpha r\frac{p_2\sqrt{1+\varphi'^2}}{\sqrt{2p_1}}$$

$$x_1 = \frac{1}{\sqrt{1+\varphi'^2}} \qquad f(x_1) = -\frac{\sqrt{2p_1}\sqrt{1+\varphi'^2}}{\varphi'}.$$

The (local) Poisson bivector found in [8] for this case reads in their coordinates (x_1, x_2, x_3, x_4) as

$$\left\{\alpha r^2\left(\frac{\partial}{\partial x_1} + \frac{f'(x_1)}{x_1}x_3\frac{\partial}{\partial x_2}\right) + mr^2\frac{x_2}{f(x_1)}\frac{\partial}{\partial x_3}\right\} \wedge \frac{\partial}{\partial x_4}$$

which, in particular, is also of the type described in example 1. Therefore, multiples of this bivector are again Poisson bivectors and hence, the rescaling introduced in [8], by means of an *invariant measure*, in order to render the reduced system Hamiltonian, is unnecessary.

On the other hand, Hermans in [25] has not noticed the existence of any of these Poisson structures of rank 2 but he constructed a closed 2-form, with domain contained in $\overline{P}_1$, which vanish in a set containing the set of regular equilibria, but has rank 4 otherwise. For this construction, which uses *non-holonomic reduction* [5], it is indeed *necessary to rescale* the original reduced vector field, see section 4.1 of [25].

4.4. Ball rolling on the interior of a cylinder

In this section, we will treat the special case of a ball rolling inside of a cylinder, which cannot be parametrized as in section 4.3. In contrast with the general case, this case is completely and explicitly solvable, as it is well known, see, e.g., [4, 8, 32, 33]. However, we will give an independent treatment.

For this specific system, we will easily find a family of Poisson structures of rank 2, generated by two of them, with respect to which the reduced system is Hamiltonian with the reduced energy as Hamiltonian function.

Consider therefore the case of the ball rolling inside a surface of revolution, with the following variations: the centre of mass of the ball will be parametrized by the vector a, with cylindrical coordinates $(\rho\cos\theta, \rho\sin\theta, z)$, where ρ is the radius of the cylinder on which the centre of mass of the ball moves, and z is the height with respect to the gravitational energy reference point. The normal vector γ reads then $\gamma = -(\cos\theta, \sin\theta, 0)$. The system (39) becomes in the coordinates $(\omega_1, \omega_2, \omega_3)$ and (θ, z)

$$\begin{aligned} \dot{\omega}_1 &= \frac{m}{\alpha}\left(\frac{g}{r} + \frac{r}{\rho}\omega_3(\omega_1\cos\theta + \omega_2\sin\theta)\right)\sin\theta \\ \dot{\omega}_2 &= -\frac{m}{\alpha}\left(\frac{g}{r} + \frac{r}{\rho}\omega_3(\omega_1\cos\theta + \omega_2\sin\theta)\right)\cos\theta \\ \dot{\omega}_3 &= 0 \qquad \dot{\theta} = -\frac{r}{\rho}\omega_3 \qquad \dot{z} = r(\omega_2\cos\theta - \omega_1\sin\theta) \end{aligned} \tag{51}$$

where $\alpha = \frac{M+mr^2}{r^2}$.

Likewise, the energy (40) reads now

$$H = \tfrac{1}{2}\{(M + mr^2)(\omega\cdot\omega) - mr^2(\omega_1\cos\theta + \omega_2\sin\theta)^2\} + mgz \tag{52}$$

which is conserved by the system (51). Obviously, ω_3 is a first integral of the system as well.

Let us consider now the system obtained after the reduction of the S^1 symmetry of rotations of the whole system about the vertical axis, as in the general case. Although now the S^1 action is free, we will use again invariant theory in order to perform the reduction. Consider the invariants similar (but not equal) to (19):

$$\begin{aligned} &\sigma_1 = z \qquad \sigma_2 = \gamma_1\omega_2 - \gamma_2\omega_1 = -\omega_2\cos\theta + \omega_1\sin\theta \\ &\sigma_3 = \gamma_1\omega_1 + \gamma_2\omega_2 = -\omega_1\cos\theta - \omega_2\sin\theta \qquad \sigma_4 = \omega_3 \end{aligned} \tag{53}$$

which in this case can be regarded as coordinates on $\mathbb{R}^4$. Then, the reduced system for $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ reads

$$\dot{\sigma}_1 = -r\sigma_2 \qquad \dot{\sigma}_2 = \frac{M\sigma_4}{\alpha r\rho}\sigma_3 + \frac{mg}{\alpha r} \qquad \dot{\sigma}_3 = -\frac{r}{\rho}\sigma_4\sigma_2 \qquad \dot{\sigma}_4 = 0 \tag{54}$$

which preserves the reduced energy

$$E = \tfrac{1}{2}\left\{mr^2\left(\sigma_2^2 + \sigma_4^2\right) + M\left(\sigma_2^2 + \sigma_3^2 + \sigma_4^2\right)\right\} + mg\sigma_1. \tag{55}$$

The reduced vector field X in the reduced space reads then

$$X = -r\sigma_2 \frac{\partial}{\partial\sigma_1} + \left(\frac{M\sigma_4}{\alpha r\rho}\sigma_3 + \frac{mg}{\alpha r}\right)\frac{\partial}{\partial\sigma_2} - \frac{r}{\rho}\sigma_4\sigma_2\frac{\partial}{\partial\sigma_3} \tag{56}$$

and we have $X(E) = 0$ in all points of $\mathbb{R}^4$. The general solution of (54) can be given explicitly. It reads

$$\begin{aligned}
\sigma_1(t) &= \sigma_1(0) - \frac{r}{\nu_1\nu_2}\left\{\sigma_2'(0)\left(1-\cos\sqrt{\nu_1\nu_2}t\right) + \sqrt{\nu_1\nu_2}\sigma_2(0)\sin\sqrt{\nu_1\nu_2}t\right\} \\
\sigma_2(t) &= \sigma_2(0)\cos\sqrt{\nu_1\nu_2}t + \frac{\sigma_2'(0)}{\sqrt{\nu_1\nu_2}}\sin\sqrt{\nu_1\nu_2}t \\
\sigma_3(t) &= -\frac{\sigma_g}{\nu_2} + \frac{\sigma_2'(0)}{\nu_2}\cos\sqrt{\nu_1\nu_2}t - \sqrt{\frac{\nu_1}{\nu_2}}\sigma_2(0)\sin\sqrt{\nu_1\nu_2}t \\
\sigma_4(t) &= \sigma_4
\end{aligned} \tag{57}$$

where we have defined the constants $\nu_1 = r\sigma_4/\rho$, $\nu_2 = M\nu_1/\alpha r^2$ and $\sigma_g = mg/\alpha r$. It is clear that the reduced system, if $\sigma_4 \neq 0$, has integral curves consisting of either periodic orbits or equilibrium points, belonging to the set $\{(\sigma_{10}, 0, -mg\rho/M\sigma_{40}, \sigma_{40}) \in \mathbb{R}^4 \mid \sigma_{10}, \sigma_{40} \in \mathbb{R}, \sigma_{40} \neq 0\}$. These equilibrium points correspond to rotations of the ball inside the cylinder at a constant height, as in the general case. On this occasion, the reduced system can be reconstructed easily to the complete system, thus the general solution of (51) is

$$\begin{aligned}
\omega_1(t) &= \sigma_2(t)\sin\theta(t) - \sigma_3(t)\cos\theta(t) \\
\omega_2(t) &= -\sigma_2(t)\cos\theta(t) - \sigma_3(t)\sin\theta(t) \\
\omega_3(t) &= \sigma_4 = \omega_3(0) \qquad z(t) = \sigma_1(t)
\end{aligned} \tag{58}$$

where $\theta(t) = \theta_0 - \nu_1 t$ and $\sigma_i(t)$, $i = 1, 2, 3$, are given by (57). If we denote $\omega_1(0) = \omega_{10}$, $\omega_2(0) = \omega_{20}$, we have the relations for the initial conditions

$$\sigma_2(0) = \omega_{10}\sin\theta_0 - \omega_{20}\cos\theta_0 \qquad \sigma_2'(0) = \sigma_g - \nu_2(\omega_{10}\cos\theta_0 + \omega_{20}\sin\theta_0).$$

The complete system, when $\omega_3 \neq 0$, is then isochronus with two frequencies, the motions being periodic (relative equilibria, projecting to equilibrium points in the reduced space) or quasi-periodic, otherwise. The solutions with $\omega_3 = 0$ correspond to *falling motions* of the ball, rolling along a vertical generatrix of the cylinder. The explicit expression of these solutions is

$$\begin{aligned}
&\omega_1(t) = \omega_{10} + t\sigma_g\sin\theta_0 \qquad \omega_2(t) = \omega_{20} - t\sigma_g\cos\theta_0 \qquad \omega_3(t) = 0 \\
&\theta(t) = \theta_0 \qquad z(t) = z_0 - \tfrac{1}{2}r\sigma_g t^2 + rt(\omega_{20}\cos\theta_0 - \omega_{10}\sin\theta_0).
\end{aligned}$$

We will treat now the question of writing the vector field X, given by (56), as a Hamiltonian vector field with respect to a Poisson structure of rank 2, with Hamiltonian function E.

We first observe that the reduced vector field X is annihilated by the 1-forms

$$\theta_1 = \mathrm{d}\sigma_4 \qquad \theta_2 = -\frac{\sigma_4}{\rho}\,\mathrm{d}\sigma_1 + \mathrm{d}\sigma_3 \qquad \theta_3 = \alpha r^2\sigma_2\sigma_4\,\mathrm{d}\sigma_2 + (M\sigma_3\sigma_4 + mg\rho)\,\mathrm{d}\sigma_3$$

and then, it is easy to apply theorem 1, to obtain the following results:

Proposition 8. *The bivector* $\Lambda_1 = \frac{\partial}{\partial\sigma_2} \wedge \frac{1}{\sigma_2}X = -\frac{\partial}{\partial\sigma_2} \wedge \left(r\frac{\partial}{\partial\sigma_1} + \frac{r\sigma_4}{\rho}\frac{\partial}{\partial\sigma_3}\right)$ *is a Poisson bivector on* $\mathbb{R}^4$ *of rank 2 such that* $\Lambda_1^\sharp(\theta_1) = \Lambda_1^\sharp(\theta_2) = 0$. *Likewise, the bivector* $\Lambda_2 = -\frac{1}{mg}\frac{\partial}{\partial\sigma_1} \wedge X = -\frac{1}{mg}\frac{\partial}{\partial\sigma_1} \wedge \left[\left(\frac{M\sigma_4}{\alpha r\rho}\sigma_3 + \frac{mg}{\alpha r}\right)\frac{\partial}{\partial\sigma_2} - \frac{r}{\rho}\sigma_4\sigma_2\frac{\partial}{\partial\sigma_3}\right]$ *is a Poisson bivector on* $\mathbb{R}^4$ *of rank 2 such that* $\Lambda_2^\sharp(\theta_1) = \Lambda_2^\sharp(\theta_3) = 0$. *In addition, we have* $X = \Lambda_1^\sharp(\mathrm{d}E) = \Lambda_2^\sharp(\mathrm{d}E)$, *where X is given by (56) and E by (55).*

Now, the Pfaff systems '$\theta_1 = 0, \theta_2 = 0$' and '$\theta_1 = 0, \theta_3 = 0$' can easily be integrated, giving non-trivial Casimir functions of Λ_1, Λ_2, and first integrals of X:

Proposition 9. *We have* $\ker \Lambda_1^\sharp = \mathrm{span}\{\mathrm{d}c_1, \mathrm{d}c_2\}$*, and* $\ker \Lambda_2^\sharp = \mathrm{span}\{\mathrm{d}c_1, \mathrm{d}c_3\}$*, where* $c_1 = \sigma_4$, $c_2 = \sigma_3 - \frac{\sigma_4}{\rho}\sigma_1$ *and* $c_3 = \frac{r\sigma_4}{\rho}\sigma_2^2 + \left(\frac{M\sigma_4}{\alpha r\rho}\sigma_3 + \frac{mg}{\alpha r}\right)\sigma_3$.

As a consequence, we have that the reduced vector field X has in principle four first integrals, namely, E, c_1, c_2 and c_3, but clearly, they form a functionally dependent set. However, for example, we have that $\{E, c_2, c_3\}$ is generically an independent set of first integrals, although in the equilibrium points one becomes dependent of the other two. In the falling motions, $\sigma_4 = 0$, therefore Λ_1 and Λ_2 become proportional.

Incidentally, we also observe that $\Lambda_1^\sharp(\mathrm{d}c_3) = \frac{1}{\alpha r^2}X$ and $\Lambda_2^\sharp(\mathrm{d}c_2) = -\frac{\sigma_4}{\rho m g}X$. In addition, the Poisson bivectors Λ_1, Λ_2 are compatible in the sense that their Schouten–Nijenhuis bracket vanishes, $[\Lambda_1, \Lambda_2] = 0$, which can be checked, e.g., using the properties (1). Thus we have the following result:

Proposition 10. *The pencil of bivectors* $\Lambda_\lambda = (1-\lambda)\Lambda_1 + \lambda\Lambda_2$ *consists of Poisson bivectors of rank 2 such that* $X = \Lambda_\lambda^\sharp(\mathrm{d}E)$ *for all* $\lambda \in \mathbb{R}$. *Moreover, the functions* $c_1, c_{2\lambda} = (1-\lambda)E - \alpha r^2 c_3$ *and* $c_{3\lambda} = \lambda r\sigma_4 E/\rho + mgrc_2$ *are (functionally dependent) Casimir functions of* Λ_λ *(and therefore, first integrals of* X*) for all* $\lambda \in \mathbb{R}$.

Proof. That the rank of Λ_λ, for all $\lambda \in \mathbb{R}$, is 2, is obvious when one realizes that it does not contain terms on $\frac{\partial}{\partial\sigma_4}$ and therefore the rank must be an even number between 0 and 4. The other statements are a matter of computation using the above observations. $\square$

5. Conclusions and outlook

We have shown the form of certain Poisson structures of rank 2 with respect to which certain reduced problems of non-holonomic mechanics become Hamiltonian. We have shown that in $\mathbb{R}^4$ and $\mathbb{R}^5$, from an algebraic point of view, these Poisson structures are defined, up to a factor function, by the choice of the kernel of bivectors on these spaces to be generated by 1-forms of a specific type. Such 1-forms define integrable codistributions in the sense of Frobenius, and are chosen in order to accommodate and generalize the systems of first order non-autonomous differential equations which appear after reduction in certain non-holonomic mechanical systems, whose solutions are related to first integrals of such reduced systems.

We have applied the theory to the cases of the rolling disc, the Routh's sphere, and the ball rolling on a surface of revolution, explicitly recovering as a particular case some results of [8]. Thus, we have shown that the framework suggested by Borisov, Mamaev and Kilin [7, 8] can be improved along the lines discussed, namely, that those reduced systems need no rescaling to become Hamiltonian with respect to a Poisson structure of rank 2, and that the domain of definition of the Poisson structures introduced therein can be extended, including even the set of singular equilibria of the reduced systems. A natural question is whether a similar approach could be used in other non-holonomic systems, maybe of higher dimension.

However, there are more fundamental points still to be better understood. For example, to what extent the mentioned Poisson structures can be useful to investigate the intimate nature of these and maybe other non-holonomic systems, for example in order to characterize their integrability properties [3, 21, 22]; see also the recent work [23]. Another question could be to clarify the origin of the system of differential equations giving the conservation laws for the mentioned reduced non-holonomic systems, see also [4, 10, 19, 32, 40] and references therein.

Acknowledgments

This work is part of the research contract HPRN-CT-2000-00113, supported by the European Commission funding for the Human Potential Research Network 'Mechanics and Symmetry in Europe' (MASIE).

The author is especially indebted to F Fassò for valuable comments, questions and ideas, concerning previous versions of this paper. Likewise, the author acknowledges useful comments and warm hospitality from J F Cariñena, R Cushman and T Ratiu at their respective institutions (Universidad de Zaragoza, University of Utrecht and EPFL).

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